Path Integral Approach to Noncommutative Quantum Mechanics

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Abstract

We consider Feynman's path integral approach to quantum mechanics with a noncommutativity in position and momentum sectors of the phase space. We show that a quantum-mechanical system with this kind of noncommutativity is equivalent to the another one with usual commutative coordinates and momenta. We found connection between quadratic classical Hamiltonians, as well as Lagrangians, in their commutative and noncommutative regimes. The general procedure to compute Feynman's path integral on this noncommutative phase space with quadratic Lagrangians (Hamiltonians) is presented. Using this approach, a particle in a constant field, ordinary and inverted harmonic oscillators are elaborated in detail.

1 INTRODUCTION

Quantum theories with noncommuting position and momentum coordinates have been investigated intensively during the recent years. Most of the re-

search has been devoted to noncommutative (NC) field theory (for a review, see e.g. [1]). Noncommutative quantum mechanics (NCQM) has been also investigated, because it can be regarded as the corresponding one-particle non-relativistic sector of NC quantum field theory and with relevance to concrete quantum systems.

In a very general NCQM one has that not only $[\hat{x_a}, \hat{p_b}] \neq 0$, but also $[\hat{x_a}, \hat{x_b}] \neq 0$ and $[\hat{p_a}, \hat{p_b}] \neq 0$. We consider here D-dimensional NCQM which is based on the following algebra:

$$[\hat{x_a}, \hat{p_b}] = i\,\hbar\,(\delta_{ab} - \frac{1}{4}\,\theta_{ac}\,\sigma_{cb}), \quad [\hat{x_a}, \hat{x_b}] = i\,\hbar\,\theta_{ab}, \quad [\hat{p_a}, \hat{p_b}] = i\,\hbar\,\sigma_{ab}, \tag{1}$$

where $\Theta = (\theta_{ab})$ and $\Sigma = (\sigma_{ab})$ are the antisymmetric matrices with constant elements.

We investigate the Feynman path integral [2] approach to NCQM,

$$\mathcal{K}(x'', t''; x', t') = \int_{x'}^{x''} \exp\left(\frac{i}{\hbar}\int_{t'}^{t''} L(\dot{q}, q, t)\,dt\right)\mathcal{D}q\,, \tag{2}$$

where $\mathcal{K}(x'', t''; x', t')$ is the kernel of the unitary evolution operator $U(t)$ and $x'' = q(t''),\ x' = q(t')$ are end points. In ordinary quantum mechanics (OQM) Feynman path integral for quadratic Lagrangians can be evaluated analytically and the exact form [3] is

$$\mathcal{K}(x'', t''; x', t') = \frac{1}{(ih)^{\frac{D}{2}}}\sqrt{\det\left(-\frac{\partial^2 \bar{S}}{\partial x''_a \partial x'_b}\right)}\exp\left(\frac{2\pi i}{h}\,\bar{S}(x'', t''; x', t')\right), \tag{3}$$

where $\bar{S}(x'', t''; x', t')$ is the action for the classical trajectory.

In this article we search the form of an effective quadratic Lagrangian which corresponds to a system with noncommutative phase space coordinates (1). This is necessary to know before to employ Feynman's path integral method in NCQM. To this end, let us note that algebra (1) of operators $\hat{x_a}$, $\hat{p_b}$ can be replaced by the equivalent one

$$[\hat{q_a}, \hat{k_b}] = i\,\hbar\,\delta_{ab}, \quad [\hat{q_a}, \hat{q_b}] = 0, \quad [\hat{k_a}, \hat{k_b}] = 0\,, \tag{4}$$

where linear transformations

$$\hat{x_a} = \hat{q_a} - \frac{\theta_{ab}\,\hat{k_b}}{2}, \quad \hat{p_a} = \hat{k_a} + \frac{\sigma_{ab}\,\hat{q_b}}{2} \tag{5}$$

are used and summation over repeated indices is assumed. According to (1), (4) and (5), NCQM related to the quantum phase space $(\hat{p},\, \hat{x})$ can be regarded as an OQM on the standard phase space $(\hat{k},\, \hat{q})$. Similar aspects of noncommutativity with $\sigma_{ab} = 0$ are considered in [4] and [5] .

2 DYNAMICS ON NONCOMMUTATIVE PHASE SPACE AND PATH INTEGRAL

Let the most general quadratic Lagrangian for a D-dimensional system be

$$L(\dot{x}, x, t) = \frac{1}{2}\, \left(\dot{x}^T\, \alpha\, \dot{x} + \dot{x}^T\, \beta\, x + x^T\, \beta^T\, \dot{x} + x^T\, \gamma\, x\right) + \delta^T\, \dot{x} + \eta^T\, x + \phi, \tag{6}$$

where coefficients of the $D\times D$ matrices $\alpha = ((1+\delta_{ab})\, \alpha_{ab}(t)),\ \beta = (\beta_{ab}(t)),\ \gamma = ((1+\delta_{ab})\, \gamma_{ab}(t))$, D-dimensional vectors $\delta = (\delta_a(t)),\ \ \eta = (\eta_a(t))$ and a scalar $\phi = \phi(t)$ are some analytic functions of the time t. Matrices α and γ are symmetric, α is nonsingular ($\det \alpha \neq 0$) and index ^{T} denotes transpose map.

The Lagrangian (6) can be presented in the more compact form:

$$L(X, t) = \frac{1}{2}\, X^T\, M\, X + N^T\, X + \phi, \tag{7}$$

where $2D \times 2D$ matrix M and $2D$-dimensional vectors X, N are defined as

$$M = \begin{pmatrix} \alpha & \beta \\ \beta^T & \gamma \end{pmatrix}, \quad X^T = (\dot{x}^T\, ,\, x^T)\, , \quad N^T = (\delta^T\, ,\, \eta^T). \tag{8}$$

Using the equations $p_a = \frac{\partial L}{\partial \dot{x}_a}$, one can express $\dot{x}$ as $\dot{x} = \alpha^{-1}\, (p - \beta\, x - \delta)$. Since $\dot{x}$ is linear in p and x, the corresponding classical Hamiltonian $H(p, x, t) = p^T\, \dot{x} - L(\dot{x}, x, t)$ becomes also quadratic, i.e.

$$H(p, x, t) = \frac{1}{2}\, \left(p^T\, A\, p + p^T\, B\, x + x^T\, B^T\, p + x^T\, C\, x\right) + D^T\, p + E^T\, x + F, \tag{9}$$

where:

$$\begin{aligned} &A = \alpha^{-1}, \qquad && B = -\,\alpha^{-1}\, \beta, \qquad && C = \beta^T\, \alpha^{-1}\, \beta - \gamma, \\ &D = -\,\alpha^{-1}\, \delta, && E = \beta^T\, \alpha^{-1}\, \delta - \eta, && F = \frac{1}{2}\, \delta^T\, \alpha^{-1} \delta\, - \phi\, . \end{aligned} \tag{10}$$

Due to the symmetry of matrices α and γ one has that matrices $A = ((1+\delta_{ab})\, A_{ab}(t))$ and $C = ((1+\delta_{ab})\, C_{ab}(t))$ are also symmetric $(A^T = A,\ C^T =$

C). The nonsingular ($\det \alpha \neq 0$) Lagrangian $L(\dot{x}, x, t)$ implies nonsingular ($\det A \neq 0$) Hamiltonian $H(p, x, t)$. Note that the inverse map, i.e. $H \to L$, is given by the same relations (10).

The Hamiltonian (9) can be also presented in the compact form

$$H(\Pi, t) = \frac{1}{2}\, \Pi^T \mathcal{M}\, \Pi + \mathcal{N}^T \,\Pi + F, \tag{11}$$

where matrix $\mathcal{M}$ and vectors Π, $\mathcal{N}$ are

$$\mathcal{M} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \quad \Pi^T = (p^T\,,\, x^T)\,, \quad \mathcal{N}^T = (D^T\,,\, E^T). \tag{12}$$

One can easily show that

$$\mathcal{M} = \sum_{i=1}^{3} \Upsilon_i^T(M)\, M\, \Upsilon_i(M), \tag{13}$$

where

$$\Upsilon_1(M) = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & -I \end{pmatrix}, \; \Upsilon_2(M) = \begin{pmatrix} 0 & \alpha^{-1}\beta \\ 0 & 0 \end{pmatrix}, \; \Upsilon_3(M) = \begin{pmatrix} 0 & 0 \\ 0 & i\sqrt{2}\, I \end{pmatrix}.$$

One has also $\mathcal{N} = Y(M)\, N$, where

$$Y(M) = \begin{pmatrix} -\,\alpha^{-1} & 0 \\ \beta^T \alpha^{-1} & -I \end{pmatrix} = -\Upsilon_1(M) + \Upsilon_2^T(M) + i\sqrt{2}\Upsilon_3(M) \tag{14}$$

and $F = N^T\, Z(M)\, N - \phi$, where

$$Z(M) = \begin{pmatrix} \frac{1}{2}\,\alpha^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\, \Upsilon_1(M) - \frac{i}{2\sqrt{2}}\, \Upsilon_3(M)\,. \tag{15}$$

Eqs. (5) can be rewritten in the compact form as

$$\hat{\Pi} = \Xi\, \hat{K}, \quad \Xi = \begin{pmatrix} I & \frac{1}{2}\,\Sigma \\ -\frac{1}{2}\,\Theta & I \end{pmatrix}, \quad \hat{K} = \begin{pmatrix} \hat{k} \\ \hat{q} \end{pmatrix}. \tag{16}$$

Since Hamiltonians depend on canonical variables, the transformation (16) lead to the transformation of Hamiltonians (9) and (11). To this end, let us quantize the Hamiltonian (9) and it easily becomes $H(\hat{p}, \hat{x}, t) = \frac{1}{2}\,(\hat{p}^T\, A\, \hat{p}$

$+ \hat{p}^T B \hat{x} + \hat{x}^T B^T \hat{p} + \hat{x}^T C \hat{x}) + D^T \hat{p} + E^T \hat{x} + F$ because (9) is already written in the Weyl symmetric form.

Performing linear transformations (5) in the above Hamiltonian we again obtain quadratic quantum Hamiltonian

$$\begin{aligned} H_{\theta\sigma}(\hat{k}, \hat{q}, t) &= \frac{1}{2} \left(\hat{k}^T A_{\theta\sigma} \hat{k} + \hat{k}^T B_{\theta\sigma} \hat{q} + \hat{q}^T B_{\theta\sigma}^T \hat{k} + \hat{q}^T C_{\theta\sigma} \hat{q} \right) \\ & + D_{\theta\sigma}^T \hat{k} + E_{\theta\sigma}^T \hat{q} + F_{\theta\sigma} , \end{aligned} \tag{17}$$

where

$$\begin{aligned} A_{\theta\sigma} &= A - \frac{1}{2} B \Theta + \frac{1}{2} \Theta B^T - \frac{1}{4} \Theta C \Theta, & D_{\theta\sigma} &= D + \frac{1}{2} \Theta E, \\ B_{\theta\sigma} &= B + \frac{1}{2} \Theta C + \frac{1}{2} A \Sigma + \frac{1}{4} \Theta B^T \Sigma, & E_{\theta\sigma} &= E - \frac{1}{2} \Sigma D, \\ C_{\theta\sigma} &= C - \frac{1}{2} \Sigma B + \frac{1}{2} B^T \Sigma - \frac{1}{4} \Sigma A \Sigma, & F_{\theta\sigma} &= F. \end{aligned} \tag{18}$$

Note that for the nonsingular Hamiltonian $H(\hat{p}, \hat{x}, t)$ and for sufficiently small θ_{ab} the Hamiltonian $H_{\theta\sigma}(\hat{k}, \hat{q}, t)$ is also nonsingular. Classical analogue of (17) maintains the same form $H_{\theta\sigma}(k, q, t) = \frac{1}{2} (k^T A_{\theta\sigma} k + k^T B_{\theta\sigma} q + q^T B_{\theta\sigma}^T k + q^T C_{\theta\sigma} q) + D_{\theta\sigma}^T k + E_{\theta\sigma}^T q + F_{\theta\sigma}$.

In the more compact form, Hamiltonian (17) is

$$\hat{H}_{\theta\sigma}(\hat{K}, t) = \frac{1}{2} \hat{K}^T \mathcal{M}_{\theta\sigma} \hat{K} + \mathcal{N}_{\theta\sigma}^T \hat{K} + F_{\theta\sigma}, \tag{19}$$

where $2D \times 2D$ matrix $\mathcal{M}_{\theta\sigma}$ and $2D$-dimensional vectors $\hat{K}$, $\mathcal{N}_{\theta\sigma}$ are

$$\mathcal{M}_{\theta\sigma} = \begin{pmatrix} A_{\theta\sigma} & B_{\theta\sigma} \\ B_{\theta\sigma}^T & C_{\theta\sigma} \end{pmatrix} , \quad \hat{K}^T = (\hat{k}^T , \hat{q}^T) , \quad \mathcal{N}_{\theta\sigma}^T = (D_{\theta\sigma}^T , E_{\theta\sigma}^T). \tag{20}$$

From (11), (16) and (19) one can find connections between $\mathcal{M}_{\theta\sigma}$, $\mathcal{N}_{\theta\sigma}$, $F_{\theta\sigma}$ and $\mathcal{M}$, $\mathcal{N}$, F, which are given by the following relations:

$$\mathcal{M}_{\theta\sigma} = \Xi^T \mathcal{M} \, \Xi, \qquad \mathcal{N}_{\theta\sigma} = \Xi^T \mathcal{N}, \qquad F_{\theta\sigma} = F. \tag{21}$$

To compute a path integral one can start from its Hamiltonian formulation on the phase space. However, such path integral on a phase space

can be reduced to the Lagrangian path integral on configuration space whenever Hamiltonian is a quadratic polynomial with respect to momentum p (see, e.g. [5]). Hence, we need the corresponding classical Lagrangians related to the Hamiltonians (17) and (19). Using equations $\dot{q}_a = \frac{\partial H_{\theta\sigma}}{\partial p_a}$ which give $k = A_{\theta\sigma}^{-1}\,(\dot{q} - B_{\theta\sigma}\,q - D_{\theta\sigma})$ we can pass from Hamiltonian (17) to the corresponding Lagrangian by relation $L_{\theta\sigma}(\dot{q}, q, t) = k^T\dot{q} - H_{\theta\sigma}(k, q, t)$. Note that coordinates q_a and x_a coincide when $\theta = \sigma = 0$. Performing necessary computations we obtain

$$\begin{aligned} L_{\theta\sigma}(\dot{q}, q, t) \;&=\; \frac{1}{2}\,\left(\dot{q}^T\,\alpha_{\theta\sigma}\,\dot{q} + \dot{q}^T\,\beta_{\theta\sigma}\,q + q^T\,\beta_{\theta\sigma}^T\,\dot{q} + q^T\,\gamma_{\theta\sigma}\,q\right) \\ &\quad + \delta_{\theta\sigma}^T\,\dot{q} + \eta_{\theta\sigma}^T\,q + \phi_{\theta\sigma}, \end{aligned} \tag{22}$$

or in the compact form:

$$L_{\theta\sigma}(Q, t) = \frac{1}{2}\,Q^T\,M_{\theta\sigma}\,Q + N_{\theta\sigma}^T\,Q + \phi_{\theta\sigma}, \tag{23}$$

where

$$M_{\theta\sigma} = \begin{pmatrix} \alpha_{\theta\sigma} & \beta_{\theta\sigma} \\ \beta_{\theta\sigma}^T & \gamma_{\theta\sigma} \end{pmatrix}, \quad Q^T = (\dot{q}^T\,,\, q^T)\,, \quad N^T = (\delta_{\theta\sigma}^T\,,\, \eta_{\theta\sigma}^T). \tag{24}$$

Then the connection between $M_{\theta\sigma}, N_{\theta\sigma}, \phi_{\theta\sigma}$ and M, N, ϕ are given by the following relations:

$$M_{\theta\sigma} = \sum_{i,j=1}^{3} \Xi_{ij}^T\;M\;\Xi_{ij}, \qquad \Xi_{ij} = \Upsilon_i(M)\;\Xi\;\Upsilon_j(\mathcal{M}_{\theta\sigma}), \tag{25}$$

and

$$N_{\theta\sigma} = Y(\mathcal{M}_{\theta\sigma})\,\Xi^T\,Y(M)\,N, \quad \phi_{\theta\sigma} = \mathcal{N}_{\theta\sigma}^T\,Z(\mathcal{M}_{\theta\sigma})\,\mathcal{N}_{\theta\sigma} - F\,. \tag{26}$$

In more detail, the connection between coefficients of the Lagrangians $L_{\theta\sigma}$ and L is given by the relations:

$$\begin{aligned} \alpha_{\theta\sigma} \;&=\; [\,\alpha^{-1} - \frac{1}{2}\,(\Theta\,\beta^T\,\alpha^{-1} - \alpha^{-1}\,\beta\,\Theta) - \frac{1}{4}\,\Theta\,(\beta^T\,\alpha^{-1}\,\beta - \gamma)\,\Theta\,]^{-1}\,, \\ \beta_{\theta\sigma} \;&=\; \alpha_{\theta\sigma}\,(\alpha^{-1}\,\beta - \frac{1}{2}\,(\alpha^{-1}\,\Sigma - \Theta\,\gamma + \Theta\,\beta^T\,\alpha^{-1}\,\beta) + \frac{1}{4}\,\Theta\,\beta^T\,\alpha^{-1}\,\Sigma)\,, \\ \gamma_{\theta\sigma} \;&=\; \gamma + \beta_{\theta\sigma}^T\,\alpha_{\theta\sigma}^{-1}\,\beta_{\theta\sigma} - \beta^T\,\alpha^{-1}\,\beta + \frac{1}{4}\,\Sigma\,\alpha^{-1}\,\Sigma \\ &\quad - \frac{1}{2}\,(\Sigma\,\alpha^{-1}\,\beta - \beta^T\,\alpha^{-1}\,\Sigma)\,, \end{aligned}$$

$$
\begin{aligned}
\delta_{\theta\sigma} &= \alpha_{\theta\sigma}\,(\alpha^{-1}\,\delta + \frac{1}{2}\,(\Theta\,\eta - \Theta\,\beta^T\,\alpha^{-1}\,\delta))\,, \\
\eta_{\theta\sigma} &= \eta + \beta^T_{\theta\sigma}\,\alpha^{-1}_{\theta\sigma}\,\delta_{\theta\sigma} - \beta^T\,\alpha^{-1}\,\delta - \frac{1}{2}\,\Sigma\,\alpha^{-1}\,\delta\,, \\
\phi_{\theta\sigma} &= \phi + \frac{1}{2}\,\delta^T_{\theta\sigma}\,\alpha^{-1}_{\theta\sigma}\,\delta_{\theta\sigma} - \frac{1}{2}\,\delta^T\,\alpha^{-1}\,\delta\,.
\end{aligned} \tag{27}
$$

Note that $\alpha_{\theta\sigma}$, $\delta_{\theta\sigma}$ and $\phi_{\theta\sigma}$ do not depend on σ.

If we know Lagrangian (6) and algebra (1) we can obtain the corresponding effective Lagrangian (22) suitable for the path integral in NCQM. Exploiting the Euler-Lagrange equations $\frac{\partial L_{\theta\sigma}}{\partial q_a} - \frac{d}{dt}\frac{\partial L_{\theta\sigma}}{\partial \dot{q}_a} = 0$ one can obtain classical trajectory $q_a = q_a(t)$ connecting end points $x' = q(t')$ and $x'' = q(t'')$, and the corresponding action $\bar{S}_{\theta\sigma}(x'', t''; x', t') = \int_{t'}^{t''} L_{\theta\sigma}(\dot{q}, q, t)\, dt$. Path integral in NCQM is a direct analogue of (3) and its exact expression in the form of quadratic actions $\bar{S}_{\theta\sigma}(x'', t''; x', t')$ is

$$
\mathcal{K}_{\theta\sigma}(x'', t''; x', t') = \frac{1}{(ih)^{\frac{D}{2}}}\sqrt{\det\left(-\frac{\partial^2 \bar{S}_{\theta\sigma}}{\partial x''_a\,\partial x'_b}\right)}\exp\left(\frac{2\,\pi\,i}{h}\,\bar{S}_{\theta\sigma}(x'', t''; x', t')\right). \tag{28}
$$

2.1 A particle in a constant field and on noncommutative phase space

The starting Lagrangian is

$$
L(\dot{x}, x) = \frac{m}{2}\,(\dot{x}_1^2 + \dot{x}_2^2) - \eta_1\,x_1 - \eta_2\,x_2. \tag{29}
$$

Using (27), one can easily find the Lagrangian $L_{\theta\sigma}(\dot{q}, q, t)$:

$$
\begin{aligned}
L_{\theta\sigma} &= \frac{m}{2}\,(\dot{q}_1^2 + \dot{q}_2^2) + \frac{\sigma}{2}\,(q_1\,\dot{q}_2 - q_2\,\dot{q}_1) + \frac{m\,\theta}{2}\,(\eta_1\,\dot{q}_2 + \eta_2\,\dot{q}_1) \\
&\quad - \left(1 - \frac{\theta\,\sigma}{4}\right)(\eta_1\,q_1 + \eta_2\,q_2) + \frac{m\,\theta^2}{8}\,({\eta_1}^2 + {\eta_2}^2)\,.
\end{aligned} \tag{30}
$$

The corresponding equations of motion are $\ddot{q}_1 - \xi\,\dot{q}_2 + \eta_1\,\phi = 0$, $\ddot{q}_2 + \xi\,\dot{q}_1 + \eta_2\,\phi = 0$, where $\xi = \frac{\sigma}{m}$, $\phi = \frac{1}{m}\left(1 - \frac{\theta\,\sigma}{4}\right)$. One can transform the above system of equations to $q_1^{(3)} + \xi^2\,\dot{q}_1 + \eta_2\,\xi\,\phi = 0$, $q_2^{(3)} + \xi^2\,\dot{q}_2 - \eta_1\,\xi\,\phi = 0$, Their solutions are: $q_1(t) = C_1 + C_2\,\cos[\,\xi\,t\,] + C_3\,\sin[\,\xi\,t\,] - \frac{\phi\,\eta_2}{\xi}\,t$, $q_2(t) = D_1 + D_2\,\cos[\,\xi\,t\,] + D_3\,\sin[\,\xi\,t\,] + \frac{\phi\,\eta_1}{\xi}\,t$, where C_1, C_2, C_3, D_1, D_2 and D_3 are

constants. Since the functions q_1 and q_2 have to satisfy coupled differential equations one has $D_2 = C_3, \quad D_3 = -\, C_2$. Employing the conditions $q_1(0) = x_1', \quad q_1(T) = x_1'', \quad q_2(0) = x_2', \quad q_2(T) = x_2''$ one obtains:

$$C_1 = \tfrac{(x_1'+x_1'')\,\xi+\phi\,\eta_2\,T}{2\,\xi} - \tfrac{\big((x_2'-x_2'')\,\xi+\phi\,\eta_1\,T\big)}{2\,\xi}\,\cot\left[\tfrac{\xi\,T}{2}\right],$$
$$C_3 = \tfrac{(x_2'-x_2'')\,\xi+\phi\,\eta_1\,T}{2\,\xi} - \tfrac{\big((x_1'-x_1'')\,\xi-\phi\,\eta_2\,T\big)}{2\,\xi}\,\cot\left[\tfrac{\xi\,T}{2}\right],$$
$$D_1 = \tfrac{(x_2'+x_2'')\,\xi-\phi\,\eta_1\,T}{2\,\xi} - \tfrac{\big((x_1''-x_1')\,\xi+\phi\,\eta_2\,T\big)}{2\,\xi}\,\cot\left[\tfrac{\xi\,T}{2}\right],$$
$$D_3 = \tfrac{(x_1''-x_1')\,\xi+\phi\,\eta_2\,T}{2\,\xi} - \tfrac{\big((x_2'-x_2'')\,\xi+\phi\,\eta_1\,T\big)}{2\,\xi}\,\cot\left[\tfrac{\xi\,T}{2}\right].$$

The Lagrangian for classical trajectory is $L_{\theta\sigma}(\dot{q},q) = -\frac{m\,\xi\,\phi\,t}{2}\Big((C_3\,\eta_1 + D_3\,\eta_2)\,\cos[\,\xi\,t\,] + (D_3\,\eta_1 - C_3\,\eta_2)\,\sin[\,\xi\,t\,]\Big) - \frac{m}{2}\Big((C_3\,D_1 - C_1\,D_3)\,\xi^2 + (\theta\,\xi + 3\,\phi)(C_3\,\eta_2 - D_3\,\eta_1)\,\cos[\,\xi\,t\,] + (C_1\,C_3 + D_1\,D_3)\,\xi^2 + (\theta\,\xi + 3\,\phi)(C_3\,\eta_1 + D_3\,\eta_2)\,\sin[\,\xi\,t\,]\Big) - \frac{m\,\phi}{2}\,(C_1\,\eta_1 + D_1\,\eta_2) + \frac{m\,(\theta\,\xi+2\,\phi)^2\,(\eta_1^2+\eta_2^2)}{8\,\xi^2}$. Using this Lagrangian we compute the classical action

$$\bar{S}_{\theta\sigma}(x'',T;x',0) = \int_0^T L_{\theta\sigma}(\dot{q},q)\,dt = \frac{m\,\xi}{2}\left(x_2''\,x_1' - x_1''\,x_2'\right) + \frac{m\,\xi\cot\left[\frac{\xi T}{2}\right]}{4}$$
$$\times\Big((x_1''-x_1')^2 + (x_2''-x_2')^2\Big) + \frac{2\,\lambda + m\,\theta\,\xi - \lambda\,\xi\,T\cot\left[\frac{\xi T}{2}\right]}{2\,\xi}$$
$$\times\Big((x_2''-x_2')\,\eta_1 - (x_1''-x_1')\,\eta_2\Big) - \frac{\lambda\,T}{2}\,\Big((x_1''+x_1')\,\eta_1 + (x_2''+x_2')\,\eta_2\Big)$$
$$+\,\frac{T}{8\,m\,\xi^2}\,\Big(2\,\xi\,\lambda^2\,T\cot\left[\frac{\xi\,T}{2}\right] + (m^2\,\theta^2\xi^2 - 4\,\lambda^2)\Big)\,(\eta_1^2+\eta_2^2)\,. \tag{31}$$

Finally, we have

$$\mathcal{K}_{\theta\sigma}(x'',T;x',0) = \frac{m\,|\xi\,|}{2\,i\,h\,|\sin\left[\frac{\xi T}{2}\right]|}\,\exp\left(\frac{2\pi i}{h}\,\bar{S}_{\theta\sigma}(x'',T;x',0)\right). \tag{32}$$

2.2 Ordinary and inverted harmonic oscillator on noncommutative plane

The Lagrangian in the question is

$$L(\dot{x},x) = \frac{m}{2}\,(\dot{x}_1^2 + \dot{x}_2^2) - \varepsilon\frac{m\,\omega^2}{2}\,(x_1^2 + x_2^2)\,, \tag{33}$$

where $\varepsilon = 1$ and $\varepsilon = -1$ are for ordinary and inverted harmonic oscillators, respectively. Using formulas (27), one can easily find the corresponding noncommutative Lagrangian

$$\begin{aligned} L_{\theta\sigma}(\dot q, q) &= \frac{m}{2\,\kappa}\,(\dot q_1^2 + \dot q_2^2) + \frac{\sigma + \varepsilon\, m^2\,\omega^2\,\theta}{2\,\kappa}\,(\dot q_2\, q_1 - \dot q_1\, q_2) \\ &- \frac{\varepsilon\, m\,\omega^2}{2\,\kappa}\,\lambda^2 ({q_1}^2 + {q_2}^2)\,, \end{aligned} \tag{34}$$

where $\kappa = 1 + \frac{\varepsilon\, m^2\,\omega^2\,\theta^2}{4}$ and $\lambda = 1 - \frac{\theta\,\sigma}{4}$.

From (34), we obtain the Euler-Lagrange equations, $\ddot q_1 - \chi\,\dot q_2 + \mu\, q_1 = 0, \quad \ddot q_2 + \chi\,\dot q_1 + \mu\, q_2 = 0$, where $\chi = \frac{\sigma}{m} + \varepsilon\, m\,\omega^2\,\theta\,, \quad \mu = \varepsilon\,\omega^2\,\lambda^2$. Let us note that these equations form a coupled system of second order differential equations, which is more complicated than in the commutative case ($\theta = \sigma = 0$). One can transform this system to $q_1^{(4)} + (\chi^2 + 2\,\mu)\, q_1^{(2)} + \mu^2\, q_1 = 0, \quad q_2^{(4)} + (\chi^2 + 2\,\mu)\, q_2^{(2)} + \mu^2\, q_2 = 0$. The solution of these equations for $|\theta\,\sigma| < 4$, $\omega > 0\,, \chi \neq 0$ and $\varepsilon > 0$ has the form $\left(\nu = \frac{\chi^2 + 4\,\mu}{4}\right)$:

$$\begin{aligned} & q_1(t) = C_1 \cos[\,\frac{\chi}{2}\,t\,] \cos[\sqrt{\varepsilon\,|\nu|}\ t\,] + C_2 \cos[\,\frac{\chi}{2}\,t\,] \Big|\sqrt{\varepsilon}\ \sin[\sqrt{\varepsilon\,|\nu|}\ t\,]\Big| \\ & + C_3 \sin[\,\frac{\chi}{2}\,t\,] \cos[\sqrt{\varepsilon\,|\nu|}\ t\,] + C_4 \sin[\,\frac{\chi}{2}\,t\,] \Big|\sqrt{\varepsilon}\ \sin[\sqrt{\varepsilon\,|\nu|}\ t\,]\Big|\,, \\ & q_2(t) = D_1 \cos[\,\frac{\chi}{2}\,t\,] \cos[\sqrt{\varepsilon\,|\nu|}\ t\,] + D_2 \cos[\,\frac{\chi}{2}\,t\,] \Big|\sqrt{\varepsilon}\ \sin[\sqrt{\varepsilon\,|\nu|}\ t\,]\Big| \\ & + D_3 \sin[\,\frac{\chi}{2}\,t\,] \cos[\sqrt{\varepsilon\,|\nu|}\ t\,] + D_4 \sin[\,\frac{\chi}{2}\,t\,] \Big|\sqrt{\varepsilon}\ \sin[\sqrt{\varepsilon\,|\nu|}\ t\,]\Big|\,. \end{aligned} \tag{35}$$

Let us note that in the case of inverted harmonic oscillator ($\varepsilon = -1$) the trigonometric functions in $\sqrt{\varepsilon\,|\nu|}\ t$ from (35) become the corresponding hyperbolic functions in $\sqrt{|\,\nu\,|}\,t$. After imposing connections between q_1 and q_2 given by coupled differential equations, we obtain the following relations between constants C and D: $D_1 = C_2, \quad D_2 = -C_1, \quad D_3 = C_4, \quad D_4 = -C_3$. The unknown constants C_1, C_2, C_3 and C_4 one can find from end conditions $q_1(0) = x_1', \quad q_1(T) = x_1'', \quad q_2(0) = x_2', \quad q_2(T) = x_2''$. Then one obtains the solutions

$$\begin{aligned} q_1 &= \frac{1}{\sin[\sqrt{\varepsilon\,|\nu|}\ T\,]} \Big(\Big(x_1''\, \cos\Big[\frac{\chi}{2}\,(t - T)\Big] + x_2''\, \sin\Big[\frac{\chi}{2}\,(t - T)\Big]\Big) \sin[\sqrt{\varepsilon\,|\nu|}\ t\,] \\ &- \Big(x_1'\, \cos\Big[\frac{\chi\, t}{2}\Big] + x_2'\, \sin\Big[\frac{\chi\, t}{2}\Big]\Big) \sin\,[\sqrt{\varepsilon\,|\nu|}\ (t - T)]\Big), \end{aligned} \tag{36}$$

$$
\begin{aligned}
q_2 \;\; = \;\; & \frac{1}{\sin[\sqrt{\varepsilon\,|\nu|}\;\,T\,]}\bigg(\Big(x_2''\,\cos\Big[\frac{\chi}{2}\,(t-T)\Big]-x_1''\,\sin\Big[\frac{\chi}{2}\,(t-T)\Big]\Big)\sin[\sqrt{\varepsilon\,|\nu|}\;\,t\,] \\
& + \Big(-x_2'\,\cos\Big[\frac{\chi\,t}{2}\Big]+x_1'\,\sin\Big[\frac{\chi\,t}{2}\Big]\Big)\sin\,[\sqrt{\varepsilon\,|\nu|}\,(t-T)\,]\bigg). \qquad (37)
\end{aligned}
$$

Inserting the above expressions and their time derivatives into (34) we find

$$
\begin{aligned}
& L_{\theta\sigma}(\dot q, q) = \frac{m\,\varepsilon\,\nu}{2\,\kappa\;\sin^2[\sqrt{\varepsilon\,|\nu|}\;T\,]}\Big((x_1''^2+x_2''^2)\,\cos\,[\,2\sqrt{\varepsilon\,|\nu|}\;t\,]+(x_1'^2+x_2'^2) \\
& \times\,\cos\,[\,2\,\sqrt{\varepsilon\,|\nu|}\;(t-T)\,]-2\cos\,[\sqrt{\varepsilon\,|\nu|}\;(2\,t-T)\,]\Big((x_1'\,x_1''+x_2'\,x_2'') \\
& \times\,\cos\,\Big[\,\frac{\chi\,T}{2}\,\Big]+(x_1''\,x_2'-x_1'\,x_2'')\,\sin\,\Big[\,\frac{\chi\,T}{2}\,\Big]\Big)\Big). \qquad (38)
\end{aligned}
$$

Using (38), we finally compute the corresponding classical action

$$
\bar S_{\theta\sigma}(x'',T;x',0) = \int_0^T L_{\theta\sigma}(\dot q, q)\,dt \qquad (39)
$$

$$
\begin{aligned}
= \;\; & \frac{m\,\sqrt{\varepsilon\,|\nu|}}{2\,\kappa\,\sin[\,\sqrt{\varepsilon\,|\nu|}\;T\,]}\Big(-\,2\,\Big((x_1'\,x_1''+x_2'\,x_2'')\cos\,\Big[\,\frac{\chi\,T}{2}\,\Big] \\
+ \;\; & (x_1'^2+x_1''^2+x_2'^2+x_2''^2)\cos\,[\,\sqrt{\varepsilon\,|\nu|}\;T\,]+2\,(-x_1''\,x_2'+x_1'\,x_2'')\sin\,\Big[\,\frac{\chi\,T}{2}\,\Big]\Big).
\end{aligned}
$$

If we take into account expression (39) then we finally have

$$
\mathcal{K}_{\theta\sigma}(x'',T;x',0) = \frac{1}{i\,h}\frac{m\,|\sqrt{\varepsilon\,|\nu|}\,|}{\kappa\,|\sin\,[\,\sqrt{\varepsilon\,|\nu|}\,T\,]|}\;\exp\left(\frac{2\pi i}{h}\,\bar S_{\theta\sigma}(x'',T;x',0)\right). \qquad (40)
$$

3 CONCLUDING REMARKS

Let us mention that taking $\sigma = 0$, $\theta = 0$ in the above formulas we recover expressions for the Lagrangian $L(\dot q, q)$, action $\bar S(x'',T;x',0)$ and probability amplitude $\mathcal{K}(x'',T;x',0)$ of the ordinary commutative case.

Note that a similar path integral approach with $\sigma = 0$ has been considered in the context of the Aharonov-Bohm effect , the Casimir effect , a quantum system in a rotating frame , and the Hall effect (for references on these and some other related subjects, see [4, 5]). Our investigation includes all systems with quadratic Lagrangians (Hamiltonians).

On the basis of the expressions presented in this article, there are many possibilities to discuss noncommutative quantum-mechanical systems with respect to various values of noncommutativity parameters θ and σ. For example, in the case $\theta\sigma = 4$ one has a critical point in the above our particular two-dimensional models (see also [6]).

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